Math 43 Midterm 2 Review

[1] Eliminate the parameter to find rectangular equations corresponding to the following parametric equations.
For [a] and [d], write y as a function of x.

[a] $x = \dfrac{t}{1-t}$, $y = \dfrac{t-1}{1+t}$

[b] $x = 3 + 5\tan t$, $y = 4 + 2\sec t$

[c] $x = 8 + 6\cos t$, $y = 7 - \sin t$

[d] $x = 5\ln 4t$, $y = 2t^3$

[2] AJ is standing 24 feet from BJ, who is 5 feet tall. AJ throws a football at 30 feet per second in BJ's direction, at an angle of 60° with the horizontal, from an initial height of 6 feet.

[a] Write parametric equations for the position of the football.
[b] Does the football hit BJ, go over BJ's head, or hit the ground before reaching BJ ?

[3] Find parametric equations for the following curves using templates from your lecture notes, textbook and exercises.

[a] the line through $(-3, -6)$ and $(7, -2)$
[b] the circle with $(-3, -6)$ and $(7, -2)$ as endpoints of a diameter
[c] the circle in [b] traversed clockwise starting at the top
[d] the ellipse with $(-3, -6)$ and $(7, -6)$ as foci, and $(2, -2)$ as one endpoint of the minor axis
[e] the hyperbola with $(-3, -6)$ and $(7, -6)$ as vertices, and $(-5, -6)$ as one focus
[f] the portion of the graph of $y = 2x^4 - 3x^3 + 1$ from $(-1, 6)$ to $(2, 9)$

[4] Find the value of $\sum_{n=3}^{8} (-1)^n n(n-4)$.

[5] Write the repeating decimal $0.4\overline{72}$ as a simplified fraction. **<u>NOTE: Only the 72 is repeated.</u>**

[6] Calculate $\dbinom{200}{4}$.

[7] Use sigma notation to write the series $\dfrac{1}{7\cdot 3} + \dfrac{1}{4\cdot 6} + \dfrac{1}{1\cdot 12} - \dfrac{1}{2\cdot 24} - \cdots - \dfrac{1}{17\cdot 768}$.

[8] Find the coefficient of x^{34} in the expansion of $(2x^5 - 3x^2)^{11}$.

[9] Find the value of $\sum_{n=3}^{\infty} 4(0.97)^{2n-1}$. **<u>HINT: Write out the first few terms first.</u>**

[10] Find the first 5 terms of the sequence defined recursively by $a_n = 2a_{n-1} - 3, \quad a_1 = 4$.
Is the sequence arithmetic, geometric or neither ? Explain how you arrived at your conclusion.

[11] Use Pascal's triangle and the Binomial Theorem to expand and simplify

[a] $(3x - 2y)^6$

[b] $\left(\sqrt{x} - \dfrac{2}{x}\right)^4$

[12] EJ bought a new car in 1998. The registration fee was $\$800$ that year. Each year, the registration fee decreased by 10%. The car was eventually sold for scrap in the year when its registration fees were $\$3.34$. What year was EJ's car sold for scrap ?

[13] CJ and DJ both just graduated from college and started new jobs. Neither could afford the market rate for apartment rentals, so they worked out deals with their landlords. CJ agreed to pay $\$400$ rent the first month, and each month after, $\$7$ more rent than the previous month. DJ agreed to pay $\$380$ rent the first month, and each month after, 2% more rent than the previous month. After 2 years, who will have paid more rent altogether, and by how much ?

[14] Prove by mathematical induction:

[a] $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all integers $n \ge 1$

[b] $\displaystyle\sum_{i=1}^{n} (2i+1)3^{i-1} = n3^n$ for all integers $n \ge 1$

[c] $a + ar + ar^2 + \cdots + ar^n = \dfrac{a(r^{n+1}-1)}{r-1}$ for all integers $n \ge 1$

[d] $\displaystyle\sum_{i=1}^{n} \frac{3}{(i+3)(i+2)} = \frac{n}{n+3}$ for all integers $n \ge 1$

[15] Find the sum of the series $-73 - 66 - 59 - 52 \cdots + 529$.

Solutions

[1] [a] $x(1-t) = t \rightarrow x - xt = t \rightarrow x = t + xt \rightarrow x = t(1+x) \rightarrow t = \dfrac{x}{1+x}$

$$y = \frac{\frac{x}{1+x} - 1}{1 + \frac{x}{1+x}} \rightarrow y = \frac{x-(1+x)}{1+x+x} \rightarrow \boxed{y = \frac{-1}{2x+1}}$$

[b] $\tan t = \dfrac{x-3}{5}$ and $\sec t = \dfrac{y-4}{2}$ and $\sec^2 t - \tan^2 t = 1 \rightarrow \boxed{\dfrac{(y-4)^2}{4} - \dfrac{(x-3)^2}{25} = 1}$

[c] $\cos t = \dfrac{x-8}{6}$ and $\sin t = 7 - y$ and $\cos^2 t + \sin^2 t = 1 \rightarrow \dfrac{(x-8)^2}{36} + (7-y)^2 = 1$

$\rightarrow \boxed{\dfrac{(x-8)^2}{36} + (y-7)^2 = 1}$

[d] $\dfrac{x}{5} = \ln 4t \rightarrow e^{\frac{x}{5}} = 4t \rightarrow t = \dfrac{1}{4}e^{\frac{x}{5}} \rightarrow y = 2\left(\dfrac{1}{4}e^{\frac{x}{5}}\right)^3 \rightarrow \boxed{y = \dfrac{1}{32}e^{\frac{3x}{5}}}$

[2] [a]
$$\begin{aligned} x &= (30\cos 60^\circ)t \\ y &= 6 + (30\sin 60^\circ)t - 16t^2 \end{aligned} \rightarrow \boxed{\begin{aligned} x &= 15t \\ y &= 6 + 15\sqrt{3}\,t - 16t^2 \end{aligned}}$$

[b] The football reaches BJ when $x = 15t = 24$ ie. when $t = 1.6$
At that time, the football's height is $y = 24\sqrt{3} - 34.96 \approx 6.61$ feet
So, the football goes over BJ's head.

[3] [a]
$$\begin{aligned} x &= -3 + (7-(-3))t \\ y &= -6 + (-2-(-6))t \end{aligned} \rightarrow \boxed{\begin{aligned} x &= -3 + 10t \\ y &= -6 + 4t \end{aligned}}$$

[b] $\text{center} = \left(\dfrac{-3+7}{2}, \dfrac{-6+(-2)}{2}\right) = (2, -4)$ $\quad$ $\text{radius} = \dfrac{1}{2}\sqrt{(7-(-3))^2 + (-2-(-6))^2} = \dfrac{\sqrt{116}}{2} = \sqrt{29}$

$$\boxed{\begin{aligned} x &= 2 + \sqrt{29}\cos t \\ y &= -4 + \sqrt{29}\sin t \end{aligned}}$$

[c] in the new timeline, the top of the circle corresponds to $s=0$ (originally $t=\frac{\pi}{2}$)

the right end of the circle corresponds to $s=\frac{\pi}{2}$ (originally $t=0$)

the bottom of the circle corresponds to $s=\pi$ (originally $t=-\frac{\pi}{2}$)

the left end of the circle corresponds to $s=\frac{3\pi}{2}$ (originally $t=-\pi$)

looking at the pairs of s and t values, we see $s+t=\frac{\pi}{2}$ or $t=\frac{\pi}{2}-s$

$$x=2+\sqrt{29}\cos(\tfrac{\pi}{2}-s) \qquad \rightarrow \qquad \boxed{x=2+\sqrt{29}\sin s}$$
$$y=-4+\sqrt{29}\sin(\tfrac{\pi}{2}-s) \qquad\qquad \boxed{y=-4+\sqrt{29}\cos s}$$

[d] $\text{center}=\left(\dfrac{-3+7}{2},-6\right)=(2,-6) \rightarrow c=7-2=5$ and $b=-2-(-6)=4$

(horizontal major axis) (vertical minor axis)

$a^2=4^2+5^2=41 \rightarrow a=\sqrt{41}$

$$\boxed{\begin{aligned}x&=2+\sqrt{41}\cos t\\ y&=-6+4\sin t\end{aligned}}$$

[e] $\text{center}=\left(\dfrac{-3+7}{2},-6\right)=(2,-6) \rightarrow c=2-(-5)=7$ and $a=7-2=5$

(horizontal transverse axis)

$b^2=7^2-5^2=24 \rightarrow b=2\sqrt{6}$

$$\boxed{\begin{aligned}x&=2+5\sec t\\ y&=-6+2\sqrt{6}\tan t\end{aligned}}$$

[f]
$$\boxed{\begin{aligned}&x=t\\ &y=2t^4-3t^3+1\\ &t\in[-1,2]\end{aligned}}$$

[4] $(-1)^3 3(3-4) + (-1)^4 4(4-4) + (-1)^5 5(5-4) + (-1)^6 6(6-4) + (-1)^7 7(7-4) + (-1)^8 8(8-4)$

$=3+0-5+12-21+32$

$=\boxed{21}$

[5] $0.4+0.072+0.00072+0.0000072+\cdots$

$$=\frac{4}{10}+\left(\frac{72}{1000}+\frac{72}{100000}+\frac{72}{10000000}+\cdots\right)$$

$$=\frac{2}{5}+\left(\frac{\frac{72}{1000}}{1-\frac{1}{100}}\right)$$

$$=\frac{2}{5}+\left(\frac{\frac{72}{1000}}{\frac{99}{100}}\right)$$

$$=\frac{2}{5}+\frac{72}{1000}\frac{100}{99}$$

$$=\frac{2}{5}+\frac{4}{55}$$

$$=\boxed{\frac{26}{55}}$$

[6] $\frac{200!}{4!\cdot 196!} = \frac{200\cdot 199\cdot 198\cdot 197\cdot 196!}{24\cdot 196!} = \boxed{64{,}684{,}950}$

[7] NOTE: The first factors in the denominator form an arithmetic sequence, and the second factors form a geometric sequence.

$$\sum_{n=1}^{9} \frac{1}{(7-3(n-1))\cdot 3(2)^{n-1}} = \boxed{\sum_{n=1}^{9} \frac{1}{3(10-3n)(2)^{n-1}}}$$

NOTE: To find the upper limit of summation, either solve $7-3(n-1)=-17$ or $3(2)^{n-1}=768$

$-3(n-1)=-24$ $\quad 2^{n-1}=256$

$n-1=8$ $\quad n-1=8$

$n=9$ $\quad n=9$

[8] The general term is $\binom{11}{r}(2x^5)^{11-r}(-3x^2)^r = \binom{11}{r}2^{11-r}(-3)^r(x^5)^{11-r}(x^2)^r = \binom{11}{r}2^{11-r}(-3)^r x^{55-3r}$

$55-3r=34 \rightarrow r=7 \rightarrow \binom{11}{7}2^{11-7}(-3)^7 = \boxed{-11{,}547{,}360}$

[9] $4(0.97)^{2(3)-1} + 4(0.97)^{2(4)-1} + 4(0.97)^{2(5)-1} + \cdots$

$= 4(0.97)^5 + 4(0.97)^7 + 4(0.97)^9 + \cdots$

$= \frac{4(0.97)^5}{1-(0.97)^2}$

$\approx \boxed{58.1207}$

[10] $a_2 = 2a_1 - 3 = 2(4)-3 = 5$

$a_3 = 2a_2 - 3 = 2(5)-3 = 7$

$a_4 = 2a_3 - 3 = 2(7)-3 = 11$

$a_5 = 2a_4 - 3 = 2(11)-3 = 19$

$\boxed{4, 5, 7, 11, 19}$

The sequence is neither arithmetic nor geometric. The differences are $1, 2, 4, 8$ which are not constant.

The ratios are $\frac{5}{4}, \frac{7}{5}, \frac{11}{7}, \frac{19}{11}$ which are also not constant.

[11] [a] $1(3x)^6(-2y)^0 + 6(3x)^5(-2y)^1 + 15(3x)^4(-2y)^2 + 20(3x)^3(-2y)^3$

$+ 15(3x)^2(-2y)^4 + 6(3x)^1(-2y)^5 + 1(3x)^0(-2y)^6$

$= \boxed{729x^6 - 2916x^5y + 4860x^4y^2 - 4320x^3y^3 + 2160x^2y^4 - 576xy^5 + 64y^6}$

[b] $1(\sqrt{x})^4\left(-\frac{2}{x}\right)^0 + 4(\sqrt{x})^3\left(-\frac{2}{x}\right)^1 + 6(\sqrt{x})^2\left(-\frac{2}{x}\right)^2 + 4(\sqrt{x})^1\left(-\frac{2}{x}\right)^3 + 1(\sqrt{x})^0\left(-\frac{2}{x}\right)^4$

$= x^2 + 4x^{\frac{3}{2}}(-2x^{-1}) + 6x(4x^{-2}) + 4x^{\frac{1}{2}}(-8x^{-3}) + 16x^{-4}$

$= \boxed{x^2 - 8x^{\frac{1}{2}} + 24x^{-1} - 32x^{-\frac{5}{2}} + 16x^{-4}}$

[12] $800(0.9)^{n-1} = 3.34 \rightarrow (0.9)^{n-1} = 0.004175 \rightarrow \ln(0.9)^{n-1} = \ln 0.004175 \rightarrow$

$(n-1)\ln 0.9 = \ln 0.004175 \rightarrow n-1 = \frac{\ln 0.004175}{\ln 0.9} \rightarrow n = 1 + \frac{\ln 0.004175}{\ln 0.9} \approx 53$

EJ's car was sold for scrap in $1998 + 53 = 2051$

[13] CJ's total rent will be $\frac{24}{2}(2 \times 400 + (24-1)(7)) = \$11{,}532$.

DJ's total rent will be $\frac{380(1.02^{24} - 1)}{1.02 - 1} = \$11{,}560.31$.

So, DJ will have paid $\$28.31$ more rent.

[14] [a] PROOF:

Basis step: $1^3 = 1 = \frac{1^2(1+1)^2}{4}$

Inductive step: Assume $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$ for some particular but arbitrary integer $k \ge 1$

Prove $1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$

$$
\begin{aligned}
&1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 \\
&= 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4}
\end{aligned}
$$

So, by mathematical induction, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for all integers $n \ge 1$

[b] PROOF:

Basis step: $\sum_{i=1}^{1}(2i+1)3^{i-1} = 3 \cdot 3^0 = 3 = 1 \cdot 3^1$

Inductive step: Assume $\sum_{i=1}^{k}(2i+1)3^{i-1} = k3^k$ for some particular but arbitrary integer $k \ge 1$

Prove $\sum_{i=1}^{k+1}(2i+1)3^{i-1} = (k+1)3^{k+1}$

$$
\begin{aligned}
&\sum_{i=1}^{k+1}(2i+1)3^{i-1} \\
&= \sum_{i=1}^{k}(2i+1)3^{i-1} + (2(k+1)+1)3^{(k+1)-1} \\
&= k3^k + (2k+3)3^k \\
&= (k + 2k + 3)3^k \\
&= (3k+3)3^k \\
&= 3(k+1)3^k \\
&= (k+1)3^{k+1}
\end{aligned}
$$

So, by mathematical induction, $\sum_{i=1}^{n}(2i+1)3^{i-1} = n3^n$ for all integers $n \ge 1$

[c] PROOF:

Basis step: $a + ar = a(1+r) = \dfrac{a(r^2 - 1)}{r-1}$

Inductive step: Assume $a + ar + ar^2 + \cdots + ar^k = \dfrac{a(r^{k+1} - 1)}{r-1}$ for some particular but arbitrary integer $k \geq 1$

Prove $a + ar + ar^2 + \cdots + ar^{k+1} = \dfrac{a(r^{k+2} - 1)}{r-1}$

$$\begin{aligned}
&a + ar + ar^2 + \cdots + ar^{k+1} \\
&= a + ar + ar^2 + \cdots + ar^k + ar^{k+1} \\
&= \frac{a(r^{k+1} - 1)}{r-1} + ar^{k+1} \\
&= \frac{a(r^{k+1} - 1) + ar^{k+1}(r-1)}{r-1} \\
&= \frac{a(r^{k+1} - 1 + r^{k+1}r - r^{k+1})}{r-1} \\
&= \frac{a(r^{k+2} - 1)}{r-1}
\end{aligned}$$

So, by mathematical induction, $a + ar + ar^2 + \cdots + ar^n = \dfrac{a(r^{n+1} - 1)}{r-1}$ for all integers $n \geq 1$

[d] PROOF:

Basis step: $\displaystyle\sum_{i=1}^{1} \frac{3}{(i+3)(i+2)} = \frac{3}{(4)(3)} = \frac{1}{4} = \frac{1}{1+3}$

Inductive step: Assume $\displaystyle\sum_{i=1}^{k} \frac{3}{(i+3)(i+2)} = \frac{k}{k+3}$ for some particular but arbitrary integer $k \geq 1$

Prove $\displaystyle\sum_{i=1}^{k+1} \frac{3}{(i+3)(i+2)} = \frac{k+1}{k+4}$

$$\begin{aligned}
&\sum_{i=1}^{k+1} \frac{3}{(i+3)(i+2)} \\
&= \sum_{i=1}^{k} \frac{3}{(i+3)(i+2)} + \frac{3}{((k+1)+3)((k+1)+2)} \\
&= \frac{k}{k+3} + \frac{3}{(k+4)(k+3)} \\
&= \frac{k(k+4)+3}{(k+4)(k+3)} \\
&= \frac{k^2+4k+3}{(k+4)(k+3)} \\
&= \frac{(k+1)(k+3)}{(k+4)(k+3)} \\
&= \frac{k+1}{k+4}
\end{aligned}$$

So, by mathematical induction, $\displaystyle\sum_{i=1}^{n} \frac{3}{(i+3)(i+2)} = \frac{n}{n+3}$ for all integers $n \geq 1$

[15] $-73 + 7(n-1) = 529 \quad \rightarrow \quad 7(n-1) = 602 \rightarrow \quad n-1 = 86 \quad \rightarrow \quad n = 87$

$S_{87} = \dfrac{87}{2}(-73 + 529) = \boxed{19{,}836}$